REAL NUMBERS

SOLUTION 7

Prove that the square of any positive integer is of the form $5q, 5q + 1, 5q + 4$ for some integer q.

Solution:

Let a be any positive integer and $b = 5$

By Euclid's division algorithm

$a = 5m + r, 0 \leq r < 5$

Possible values of $r = 0, 1, 2, 3, 4$

When $r = 0, a = 5m$

Squaring both sides

$a^2 = (5m)^2 = 25m^2 = 5(5m^2) = 5q$, where $q = 5m^2$

When $r = 1, a = 5m + 1$

Squaring both sides

$a^2 = (5m + 1)^2$

$= (5m)^2 + 2(5m)(1) + (1)^2$

$= 25m^2 + 10m + 1$

$= 5(5m^2 + 2m) + 1$

$= 5q + 1$, where $q = 5m^2 + 2m$

When $r = 2, a = 5m + 2$

Squaring both sides

$a^2 = (5m + 2)^2$

$= (5m)^2 + 2(5m)(2) + (2)^2$

$= 25m^2 + 20m + 4$

$= 5(5m^2 + 4m) + 4 = 5q + 4$, where $q = 5m^2 + 4m$

When $r = 3, a = 5m + 3$

Squaring both sides

$a^2 = (5m + 3)^2$

$= (5m)^2 + 2(5m)(3) + (3)^2$

$= 25m^2 + 30m + 9$

$= 25m^2 + 30m + 5 + 4$

$= 5(5m^2 + 6m + 1) + 4 = 5q + 4$, where $q = 5m^2 + 6m + 1$

When $r = 4, a = 5m + 4$

Squaring both sides

$a^2 = (5m + 4)^2$

$= (5m)^2 + 2(5m)(4) + (4)^2$

$= 25m^2 + 40m + 16$

$= 25m^2 + 40m + 15 + 1$

$= 5(5m^2 + 8m + 3) + 1 = 5q + 1$, where $q = 5m^2 + 8m + 3$

$\therefore$ The square of any positive integer is of the form $5q, 5q + 1, 5q + 4$ for some integer q.